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Optimal Covering of Plane Domains by Circles Via Hyperbolic Smoothing

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Abstract. We consider the problem of optimally covering plane domains by a given number of circles. The mathematical modeling of this problem leads to a min–max–min formulation which, in addition to its intrinsic multi-level nature, has the significant characteristic of being non-differentiable. In order to overcome these difficulties, we have developed a smoothing strategy using a special class C^∞ smoothing function. The final solution is obtained by solving a sequence of differentiable subproblems which gradually approach the original problem. The use of this technique, called Hyperbolic Smoothing, allows the main difficulties presented by the original problem to be overcome. A simplified algorithm containing only the essential of the method is presented. For the purpose of illustrating both the actual working and the potentialities of the method, a set of computational results is presented.

Key words: Location problems, Min–max–min problems, Non-differentiable programming, Smoothing

1. Introduction

Let S be a finite region in $\Re^2$. A set of q figures constitutes a covering of order 1 of S if each point $s \in S$ belongs to at least one figure. Coverages of a higher order can be defined in a similar manner.

Problems inherent to the covering of $\Re^2$ regions by circles, of $\Re^3$ regions by spheres, and even regions in higher dimensional spaces have been the object of research for many decades. Important results in the study of these problems appear in Rogers (1964), Toth (1964), Toth (1983), Conway and Sloane (1988) and Hales (1992). The covering of plane domains by a set of ellipses was studied by Galiyev (1995).

In this paper, we consider the special case of covering a finite plane domain S optimally by a given number q of circles $C_i = 1, \dots, q$. This problem arises in a large number of practical applications, such as: solving some crystallography models, placing service centers, or locating and dimensioning telecommunications centers. The study of the latter topic, incidentally, provided the initial motivation for the development of this work.

The core focus of this paper is the smoothing of the min–max–min problem engendered by the modeling of the covering problem. In a sense, the process whereby this is achieved is an extension of a smoothing scheme, called Hyperbolic Smoothing, presented in Santos (1997) for non-differentiable problems in general and in Chaves (1997) for the min–max problem. This technique was developed through an adaptation of the hyperbolic penalty method originally introduced by Xavier (1982).

By smoothing we fundamentally mean the substitution of an intrinsically non-differentiable three-level problem by a differentiable single-level alternative. This is achieved through the solution of a sequence of differentiable problems which gradually approach the original problem.

Several researchers have presented alternative smoothing methods. The solution of the finite min–max problem, generally non-differentiable, has been a special motivation for the development of those methods, such as, for example, the strategies indicated in Bertsekas (1982), Polyak (1988), Pillo et al. (1993), Pinar and Zenios (1994) and Galiyev (1997). An extensive survey on these problems can be found in Du and Pardalos (1995).

This work is organized in the following way. We begin with a detailed introduction to the covering problem in Section 2. The new methodology is described in Section 3. The algorithm and the illustrative computational results are presented in Sections 4 and 5. We then conclude in Section 6.

2. The Covering Problem as a Min–Max–Min Problem

In order to formulate the original covering problem as a min–max–min problem, we proceed as follows. Let $x_i, i = 1, \ldots, q$ be the centers of the circles that must cover a domain $S \subseteq \Re^2$. The set of these center coordinates will be represented by $X \in \Re^{2q}$. Given a point s of S, we initially calculate the distance from s to the center in X that is nearest. This is given by

$$d(s, X) = \min_{x_i \in X} \|s - x_i\|_2. \tag{1}$$

Distance $d(s, X)$ provides a measurement of the covering for a specific point $s \in S$. A measurement of the quality of a covering of domain S by the q circles is, of course, provided by the largest distance $d(s, X)$, which corresponds exactly to the most critical covering of a point. Letting $D(X)$ denote this largest distance, we have

$$D(X) = \max_{s \in S} d(s, X). \tag{2}$$

The optimal placing of the centers must provide the best-quality covering for S, that is, it must minimize the most critical covering. If X^* denotes an optimal placement, then

$$X^* = \operatorname*{argmin}_{X \in \Re^{2q}} D(X), \tag{3}$$

where X is the set of all placements. Using (1)–(3), we finally arrive at

$$X^* = \operatorname*{argmin}_{X \in \Re^{2q}} \max_{s \in S} \min_{x_i \in X} \|s - x_i\|_2. \tag{4}$$

3. Transforming the Problem

In order to solve (4) numerically, we first discretize the domain S into a finite set of m points $s_j, j = 1, \dots, m$, thus obtaining

$$X^* = \operatorname*{argmin}_{X \in \Re^{2q}} \max_{j=1,\dots,m} \min_{x_i \in X} \|s_j - x_i\|_2 \tag{5}$$

If for fixed j, we let $z_j(X)$ denote the innermost minimum in (5), that is

$$z_j(x) = \min_{x_i \in X} \|s_j - x_i\|_2, \tag{6}$$

then, $z_j(x)$ must necessarily satisfy the following set of inequalities:

$$z_j(x) - \|s_j - x_i\|_2 \leqslant 0, \quad i = 1, \dots, q. \tag{7}$$

Similarly, if $z(X)$ denotes the maximum in (5) for fixed X, that is

$$z(X) = \max_{j=1,\dots,m} z_j(X), \tag{8}$$

then $z(X)$ is required to satisfy the constraints

$$z(X) \geqslant z_j(X), \quad j = 1, \dots, m. \tag{9}$$

and the solution of the outermost level of problem (5), is not altered if we dissociate z and X thus obtaining the equivalent problem

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{subject to } z_j = \min_{i=1,\dots,q} \|s_j - x_i\|_2, \quad j = 1, \dots, m \\ &\qquad\qquad z \geqslant z_j, \quad j = 1, \dots, m. \end{aligned} \tag{10}$$

Now, consider the similar optimization problem on the same variables $z, z_1, \dots, z_m, x_1, \dots, x_q$

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{subject to} \quad z_j - \|s_j - x_i\|_2 \leqslant 0, \quad j = 1, \dots, m, \quad i = 1, \dots, q \\ &\qquad\qquad z \geqslant z_j, \quad j = 1, \dots, m. \end{aligned} \tag{11}$$

This problem is not equivalent to (10) since the variables z_j are not bounded from below, so neither is z. In order to obtain the desired equivalence we must, therefore, modify problem (11). We do so by first letting $\varphi(y)$ denote $\max\{0, y\}$ and then observing that from the first set of inequalities in (11), it follows that

$$\sum_{i=1}^{q} \varphi(z_j - \|s_j - x_i\|_2) = 0, \quad j = 1, \ldots, m. \tag{12}$$

For fixed j and assuming $d_1 < \cdots < d_q$ with $d_i = \|s_j - x_i\|_2$, Figure 1 illustrates the first three summands of (12) as a function of z_j.

Using (12) in place of the first set of inequalities in (11) we obtain an equivalent problem maintaining, therefore, the undesirable property that z and z_j still have no lower bound. Considering, however, that the objective function of problem (11) will force z and, consequently, each z_j, $j = 1, \ldots, m$, downward, we can think of bounding the latter variables from below by considering ">" in place of "=" in (12) and considering the resulting 'non-canonical' problem

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{subject to} \quad \sum_{i=1}^{q} \varphi(z_j - \|s_j - x_i\|_2) > 0, \quad j = 1, \ldots, m. \\ &\qquad\qquad\quad z \geqslant z_j, \quad j = 1, \ldots, m. \end{aligned} \tag{13}$$

The canonical formulation can be recovered from (13) by perturbing (12) and considering the modified problem:

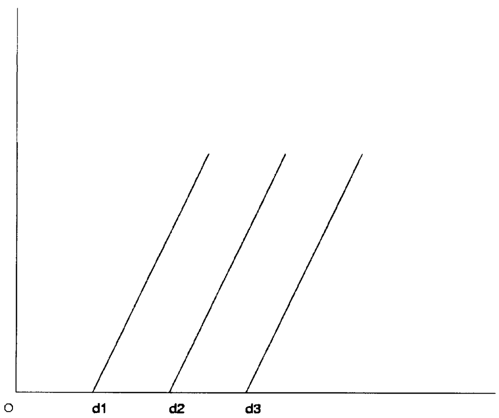

Figure 1. Summands in (12).

$$
\begin{aligned}
&\text{minimize} \quad z \\
&\text{subject to} \quad \sum_{i=1}^{q} \varphi(z_j - \|s_j - x_i\|_2) \geqslant \varepsilon, \quad j = 1, \dots, m, \\
&\qquad\qquad\quad z \geqslant z_j, \quad j = 1, \dots, m
\end{aligned}
\tag{14}
$$

for $\varepsilon > 0$. Since the feasible set of problem (13) is the limit of that of (14) when $\varepsilon \to 0_+$, we can then consider solving (13) by solving a sequence of problems like (14) for a sequence of decreasing values for ε that approaches 0.

We can now present the main theoretical results that support our methodology for covering a plane domain optimally.

THEOREM 1. *Problems (10) and (13) have the same optimal value.*

Proof. Let z' be the optimum value of problem (13) and $(z'', (z''_j, j = 1, \dots, m), (x''_i, i = 1, \dots, q))$ an optimum solution of problem (10). Also consider for any $j \in \{1, \dots, m\}$, the set

$$I_j \triangleq \{1 \in \{i, \dots, q\} | z''_j = \|s_j - x''_i\|_2\}.$$

Now observe that, for any $\varepsilon > 0$, $(z'' + \varepsilon, (z''_j + \varepsilon,\ j = 1, \dots, m), (x''_i, i = 1, \dots, q))$ is a feasible solution of (13) since:

$$
\begin{aligned}
\sum_{i=1}^{q} \varphi(z''_j + \varepsilon - \|s_j - x''_i\|_2) &= \sum_{\{i | z''_j + \varepsilon \geqslant \|s_j - x''_i\|_2\}} (z''_j + \varepsilon - \|s_j - x''_i\|_2) \\
&\geqslant \sum_{\{i \in I_j\}} (z''_j + \varepsilon - \|s_j - x''_i\|_2) = \#(I_j).\ \varepsilon \geqslant \varepsilon > 0
\end{aligned}
$$

for any $j \in \{1, \dots, m\}$ and

$$z'' + \varepsilon \geqslant z''_j + \varepsilon,\ j = 1, \dots, m.$$

Thus, for every $\varepsilon > 0$ (13) has a feasible solution where its objective function values $z'' + \varepsilon$ which implies that $z' \leqslant z''$.

On the other hand, for any feasible solution $(\tilde{z}, (\tilde{z}_j, j = 1, \dots, m), (\tilde{x}_i, i = 1, \dots, q))$ of (13) there is $\tilde{i} \in \{1, \dots, q\} \,|\, \tilde{z}_j > \|s_j - \tilde{x}_{\tilde{i}}\|_2$, otherwise $\Sigma_{i=1}^{q} \varphi(\tilde{z}_j - \|s_j - \tilde{x}_i\|_2) = 0$. This is equivalent to say that $\tilde{z}_j > \min_i \ (\|s_j - \tilde{x}_i\|_2)$ and as $\tilde{z} > \tilde{z}_j, \forall_j \in \{1, \dots, m\}$, the following sequence of relations holds:

$$\tilde{z} \geqslant \max_j \tilde{z}_j \geqslant \max_j \quad \min_i \|s_j - \tilde{x}_i\|_2 \geqslant \min_{(x_i, i=1,\dots,q)} \quad \max_j \quad \min_i \|s_j - x_i\|_2 = z''.$$

Therefore, z'' is a lower bound for the value of z at any feasible solution of (13) which immediately leads to $z' \geqslant z''$. As the opposite inequality is also valid, z' and z'' must be equal, completing the proof. □

Henceforth in this section we let $z', z'_1 \dots, z'_m, x'_1, \dots, x'_q$ be an optimal solution of problem (13). Because φ is non-decreasing and $z' \geqslant z'_j$ for $j = 1, \dots, m$, we have

$$\sum_{i=1}^{q} \varphi(z' - \|s_j - x'_i\|_2) \geqslant \sum_{i=1}^{q} \varphi(z' - \|s_j - x'_i\|_2) > 0, \quad j = 1, \dots, m. \tag{15}$$

We may then substitute z' for each of $z'_1, \dots, z'_m$ and still have a feasible solution of (13) that is obviously optimal, since we have not changed the solution value. This observation constitutes Preposition 1, given next, which allows a considerable reduction in the problem's dimension by the elimination of variables $z'_l, \dots, z'_m$.

PROPOSITION 1. *Problem (13) admits at least one optimal solution for which* $z'_j = z', j = 1, \dots, m$.

Now, consider the problem:

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{subject to} \quad \sum_{i=1}^{q} \varphi(z - \|s_j - x_i\|_2) > 0, \quad j = 1, \dots, m. \end{aligned} \tag{16}$$

Problem (13) is the limit of (14) when $\varepsilon \to 0_+$. A similar observation is valid for problem (16).

It must be emphasized that problem (16) is defined in a $(2q+1)$-*dimensional* space, much smaller, therefore, than the solution space of problems (10) and (13), whose dimension is $(2q + m + 1)$. Thus solving (16) in place of problem (13), if possible, will be certainly computationally advantageous. Proposition 1 leads exactly to the equivalence between the two problems, which constitutes Theorem 2, given next.

THEOREM 2. *Problems (16) and (13) are equivalent to each other.*

Proof. If $x^*, z^*, z^*_j, j = 1, \dots, m$ is an optimum solution of problem (13) then x^*, z^* is a feasible point for problem (16), since, due to the definition of the function $\varphi(.)$, satisfying the constraints of (13) implies that those of (16) are also satisfied.

Conversely, let $\hat{x}, \hat{z}$ be an optimal solution of problem (16). Defining:

$$\hat{z}_j = \hat{z}, \quad j = l, \dots, m, \tag{17}$$

$\hat{x}, \hat{z}, \hat{z}_j,\ j = l, \dots, m$ will be a feasible point for problem (13). □

4. Smoothing the Problem

Although problem (16) has a reduced dimension, the definition of function φ endows it with an extremely rigid non-differentiable structure, which

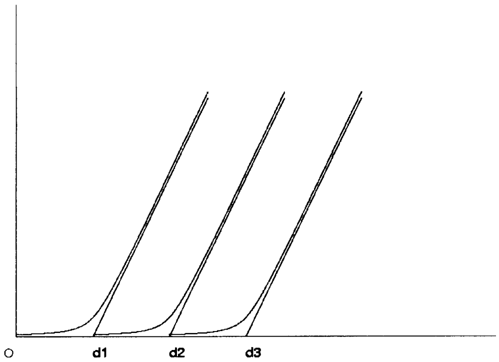

Figure 2. Original and smoothed summands in (12).

makes its computational solution very hard. In view of this, the numerical method we adopt for solving problem (16), takes a smoothing approach. From this perspective, let us define the function:

$$\phi(y,\tau) = \left(y + \sqrt{y^2 + \tau^2}\right)/2 \tag{18}$$

for $y \in \Re$ and $\tau > 0$.

Function ϕ constitutes an approximation of function φ. Adopting the same assumptions used in Figure 1, the first three summands of (12) and their corresponding smoothed approximations, given by (18), are depicted in Figure 2.

In addition, ϕ has the following properties:
(a) $\phi(y,\tau) > \varphi(y), \quad \forall\tau > 0$;
(b) $\lim_{\tau\to 0} \phi(y,\tau) = \varphi(y)$;
(c) $\phi(.,\tau)$ is an increasing convex C^∞ function.

These properties allows us to seek a solution to problem (16) by solving a sequence of subproblems of the form

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{subject to} \quad \sum_{i=1}^{q} \phi(z - \|s_j - x_i\|_2, \tau) \geqslant \varepsilon, \quad j = 1, \ldots, m. \end{aligned} \tag{19}$$

This process constitutes the Hyperbolic Smoothing Algorithm, described below in a simplified form.

Simplified Algorithm

Initialization Step. Choose values $0 < \rho_2 \leqslant \rho_1 < 1$; let $k = 1$ and choose initial values: x^0, ε^1 and τ^1.

Main Step. Repeat indefinitely

Solve problem (19) with $\tau = \tau^k$ and $\varepsilon = \varepsilon^k$, starting at the initial point x^{k-1}, and let x^k be the solution obtained.

Let $\tau^{k+1} = \rho_1 \tau^k, \varepsilon^{k+1} = \rho_2 \varepsilon^k$, and $k := k+1$.

Just as in other smoothing methods, the solution to the covering problem is obtained by resolving an infinite sequence of constrained minimization subproblems ($k = 1, 2, \ldots$ in the Main Step).

Notice that the algorithm causes τ and ε approach 0, so the constraints of the subproblems it solves, given as in (19), tend to those of (16). Also, the algorithm assumes that x^k is a global solution to the k^{th} smoothed subproblem it solves. Under this hypothesis, and owing to the continuity properties of all functions involved, the sequence $z^1, z^2, \ldots$ of optimal values tends to the optimal value of (16). Moreover, as $\rho_2 \leqslant \rho_1$, the optimal solution of a subproblem is feasible for the next one, thus causing the optimal values to decrease monotonically, since the objective function is always the same.

5. Computational Results

In order to illustrate the functioning of the method, we present some computational results on two small synthetic test instances, whose optimal solutions are known beforehand. They have been created in order to perform a preliminary validation of the method.

Figures 3 and 4 show graphic depictions of the two instances. The first one is the domino instance, consisting of 6 points to be covered by 2 circles. The second is the rosaceous instance with 4 petals, where 24 points must be covered by 4 circles. The solutions obtained were identical to the exact solutions to the 15th decimal place. In addition, we obtained results of similar quality solving rosaceous problems with 8, 12, 16, 24, 32, and 48 petals, with 6 points by petal in each one.

Table 1 shows the sequence of points generated by the method in solving the first instance. Columns k and τ^k represent the iteration and the value of the smoothing parameter, while the pair (a_1^k, b_1^k) represents the coordinates of the center of the first circle x_1^k, the pair (a_2^k, b_2^k) the coordinates of the center of the second circle, and z^k the radius of the circles. The results obtained in the computational experiments with problems of larger dimension showed analogous behavior.

Figures 5–7 show the computational results obtained in the solution of six real covering problems: The Netherlands (5 circles), Brazil (5 circles), the Brazilian states of Ceará (8 circles) and Rio de Janeiro (9 circles), the state of New York (7 circles) and the USA (5 circles). The number of discretization points were, respectively, 9220, 6620, 3160, 3539, 7225, and 4752. The execution time in seconds were 109.02, 93.87, 78.55, 70.52,

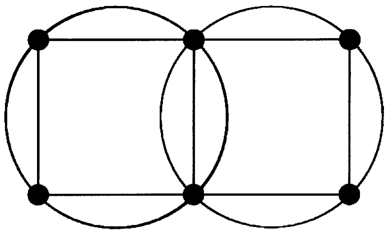

Figure 3. The first instance.

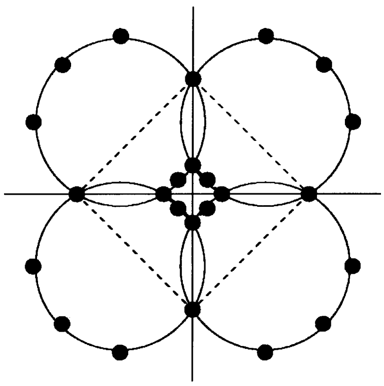

Figure 4. The second instance.

Table 1. Sequence of points generated in solving the first instance

k	τ^k	a_1^k	b_1^k	a_2^k	b_2^k	z^k
1	0.1E–1	0.4934416	0.5003126	1.506524	0.4999621	0.7148618
2	0.1E–2	0.4976222	0.4976221	1.502381	0.5023814	0.7074767
3	0.1E–3	0.4994190	0.4994190	1.500581	0.5005810	0.7071485
4	0.1E–4	0.4998605	0.4998605	1.500139	0.5001395	0.7071115
5	0.1E–5	0.4999668	0.4999668	1.500033	0.5000332	0.7071073
6	0.1E–6	0.4999921	0.4999921	1.500008	0.5000079	0.7071068
7	0.1E–7	0.4999981	0.4999981	1.500002	0.5000019	0.7071068
8	0.1E–8	0.4999996	0.4999996	1.500000	0.5000004	0.7071068
9	0.1E–9	0.5000000	0.5000000	1.500000	0.5000000	0.7071068
Solution	–	0.5000000	0.5000000	1.500000	0.5000000	0.7071068

116.17, and 74.04, respectively. Problem (19) was solved by the Hyperbolic Penalty Method (cf. Xavier, 1982). In this method, the solution is obtained by solving a sequence of unconstrained subproblems. The unconstrained minimizations were carried out by means of a Quasi-Newton algorithm employing the BFGS updating formula from the Harwell Library. The execution times reported were obtained on an Intel 486 system running at 150 MHz. The precision of all solutions was limited by the resolution adopted for the meshes.

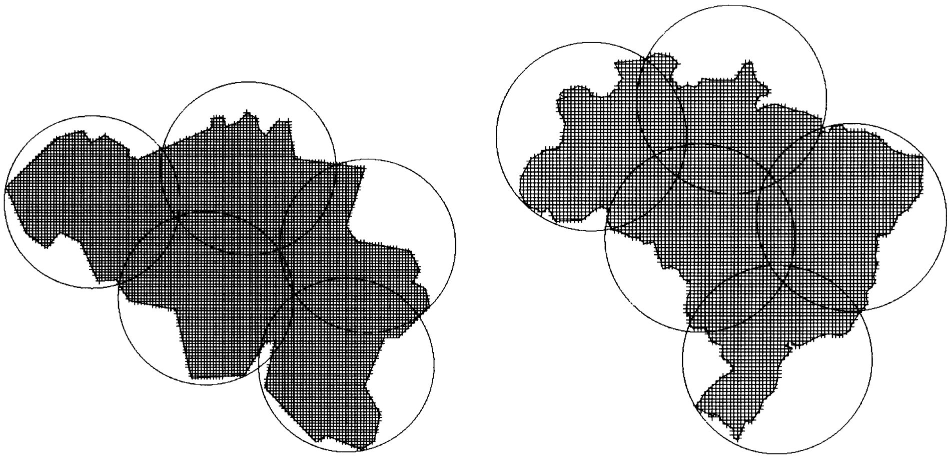

Figure 5. Coverages of the Netherlands and of Brazil.

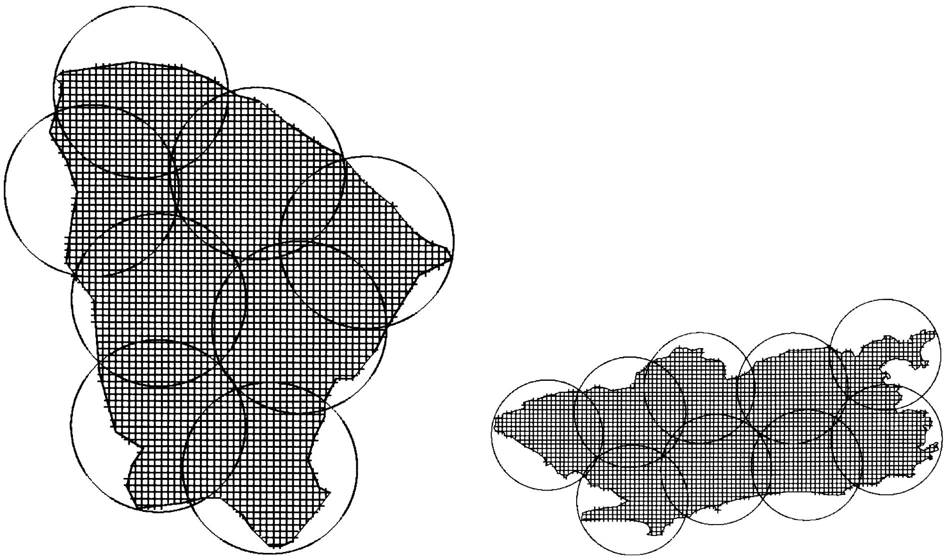

Figure 6. Coverages of the Brazilian states of Ceara and Rio de Janeiro.

6. Conclusions

In view of the results obtained, where the proposed algorithm performed efficiently and robustly in accordance to the theory developed, we believe that the algorithm can be used to solve large, practical optimal covering problems.

Moreover, it must be observed that the methodology introduced in this article can be applied to any min–max–min problem. Among them, we con-

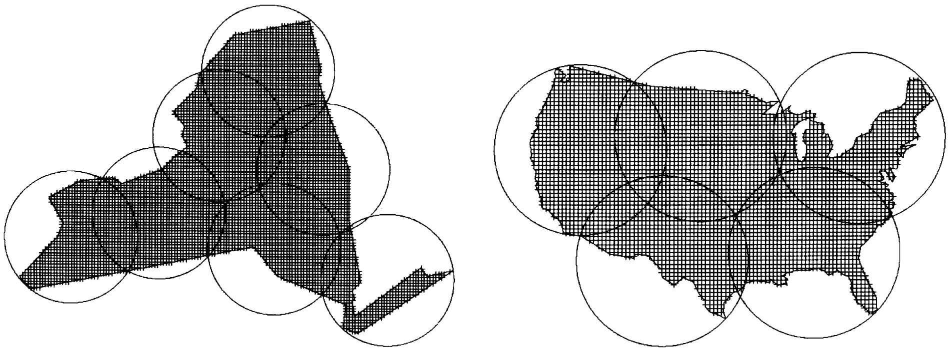

Figure 7. Coverages of the state of New York and of the USA.

sider it to be particularly interesting to try this approach on the problem of controlling the water level of a power plant reservoir (cf. Demyanov, 1971).

A similar problem is the one of locating base radio stations (The BRS problem) so that any site in a region S can be reached, an application that emerged in the area of cellular telephony. The sketch proposed in this paper can be used to obtain a first approximation to that problem, and can also be adapted to approach several of its variations, such as, the one in which a subset of stations already exists and another in which the *loci* of the stations are constrained to lie in a subset of S.

There are several possibilities for the continuation of this work. One alternative to be explored is to make the mesh points, that must be covered at an iteration of the algorithm, be chosen in function of the preceding iteration solution in a way that considerably reduces the number of them. Such a reduction would make it possible to consider schemes in which, instead of working with a constant discretization of the region S, we could increase the precision of the grid at every iteration. Each new grid, however, would be constrained to the subset of S consisting of points y satisfying $\min_{i=1,\ldots,q} \|y - x_i^{k-1}\|_2 > r\,z^{k-1}$, where r is a number less than 1 but close to 1.

Another variant to be tested is to append to the method a final phase comprising a linear search algorithm. We recall that, if the region S to be covered is polygonal, as in most practical cases, then the function $F : s_i, i = 1, \ldots, q \longrightarrow \max_{y \in S} \min_{i=1,\ldots,p} \|y - s_i\|$ has directional derivatives for all directions in $\Re^{2q}$ (cf. Oliveira, 1979). Such derivatives are also max–min functions whose computational complexity is dependent not only on the number of the covering elements q but also on the amount of points in S that are 'worst covered' by $x_i, i = 1, \ldots, q$, and on how many of these points are on the region's border. Computing them, however, has the same complexity as calculating F, which makes the use of linear search methods to obtain optimal coverings by circles an option to be considered. Finally we must remark that the methodology described in this work can be easily adapted to similar problems in the contexts of covering, packing(Maranas et al., 1995) and clustering.

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